Acyclic Coloring of Graphs

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ABSTRACT

A vertex coloring of a graph G is called *acyclic* if no two adjacent vertices have the same color and there is no two-colored cycle in G. The *acyclic chromatic number* of G, denoted by $A(G)$, is the least number of colors in an acyclic coloring of G. We show that if G has maximum degree d, then $A(G) = O(d^{4/3})$ as $d \rightarrow \infty$. This settles a problem of Erdös who conjectured, in 1976, that $A(G) = o(d^2)$ as $d \rightarrow \infty$. We also show that there are graphs G with maximum degree d for which $A(G) = \Omega(d^{4/3}/(\log d)^{1/3})$; and that the *edges* of any graph with maximum degree d can be colored by $O(d)$ colors so that no two adjacent edges have the same color and there is no two-colored cycle. All the proofs rely heavily on probabilistic arguments.

1. INTRODUCTION

All graphs considered here are finite, undirected and have no loops and no multiple edges. A vertex coloring of a graph $G = (V, E)$ is *acrylic* if it is a proper vertex coloring (that is, adjacent vertices have distinct colors), and there is no cycle in the subgraph induced by the vertices of any two of the colors. The *acyclic chromatic number* of G, denoted by $A(G)$, is the least number of colors in an acyclic coloring of G.

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Acyclic colorings were introduced by Grunbaum [8] and studied by Albertson and Berman [1] and by Borodin [6] amongst others. Most of the published results about acyclic colorings consider graphs drawn on some fixed surface. Perhaps for any surface the maximum acyclic chromatic number equals the maximum usual chromatic number (except for the sphere where the numbers are 5 and 4, respectively). This problem was raised in [1] and also by Borodin. A.V. Kostochka proved in 1978 in his thesis that it is an NP-complete problem to decide for a given G and k if the acyclic chromatic number of G is at most k.

Let $\Delta = \Delta(G)$ denote the maximum degree of a vertex of a graph G. By coloring sequentially the vertices of G, where each vertex v, in its turn, is colored by the first color which is not assigned already to a vertex at distance at most two from v, one easily obtains an acyclic coloring of G with at most $\Delta^2 + 1$ colors. For $d = 1, 2, \ldots$ define

$$A(d) = \max\{A(G): \Delta(G) = d\} .$$

By the above remarks $A(d) \leq d^2 + 1$. In 1976 Erdös conjectured that $A(d) = o(d^2)$ as $d \to \infty$ (see [1] and [11, problem 37]). This conjecture is established by the following theorem.

Theorem 1.1.

$$A(d) = O(d^{4/3}) .$$

The estimate in the above theorem is not far from the truth, as shown by the next theorem (We shall use natural logarithms throughout.)

Theorem 1.2.

$$A(d) = \Omega\left(\frac{d^{4/3}}{(\log d)^{1/3}}\right) .$$

It is noted in [1] that Erdös had showed that $A(d) = \Omega(d^{4/3-\epsilon})$. From these results we know that there are graphs with maximum degree d whose acyclic chromatic number is significantly larger than d. It turns out that such graphs must contain certain complete bipartite graphs. Let $K_{a,b}$ denote the complete bipartite graph with vertex classes of sizes a and b.

Theorem 1.3. *Let G be a graph with maximum degree $d \geq 1$ and suppose that for some $\gamma \geq 1$, G contains no copy of $K_{2,\gamma+1}$ in which the two vertices in the first class are nonadjacent. Then*

$$A(G) = O(\sqrt{\gamma}d) .$$

In particular, if the girth of G is at least 5, then $A(G) = O(d)$. Another interesting special case of Theorem 1.3, which follows from the fact that line graphs contain no copy of $K_{2,5}$ in which the two vertices of the first class are nonadjacent, is the following.

Corollary 1.4. *The edges of any graph G with maximum degree d can be colored by $O(d)$ colors such that no two adjacent edges have the same color and there is no cycle in the subgraph containing the edges of any two of the colors.*

The proofs of all the three theorems above rely heavily on probabilistic arguments. Theorems 1.1 and 1.3 are proved by applying the Erdös–Lovász local lemma first proved in [7] (see also [4] and [10]). For some recent applications of this lemma to various other decomposition problems see [2] and [3].

The paper is organized as follows. In Section 2 we consider graphs with maximum degree d and prove Theorems 1.1 and 1.2. Section 3 contains the proof of Theorem 1.3 (and that of Corollary 1.4). The final Section 4 contains some concluding remarks and open problems.

2. GRAPHS WITH MAXIMUM DEGREE *d*

In order to prove Theorem 1.1 we need the Erdös–Lovász local lemma (in its nonsymmetric form), which is the following.

Lemma 2.1 ([7], see also [10]). *Let $A_1, A_2, \dots, A_n$ be events in an arbitrary probability space. Let the graph $H=(V,E)$ on the nodes $\{1,2,\dots,n\}$ be a **dependency graph** for the events A_i; that is, assume that for each i, A_i is independent of the family of events $\{A_j : \{i,j\} \not\in E\}$. If there are reals $0 \le y_i < 1$ such that for all i*

$$\Pr(A_i) \le y_i \prod_{\{i,j\}\in E} (1-y_j)$$

then

$$\Pr(\bigcap_i \bar{A}_i) \ge \prod_{i=1}^{n} (1-y_i) > 0 \,,$$

so that with positive probability no event A_i occurs. ∎

We actually prove the following explicit version of Theorem 1.1.

Proposition 2.2. *Let $G=(V,E)$ be a graph with maximum degree d. Then $A(G) \le \lceil 50d^{4/3} \rceil$.*

Remark 2.3. The constant 50 can be easily improved. We make no attempt to optimize the constants here, and throughout the article.

Proof of Proposition 2.2. Put $x = \lceil 50d^{4/3} \rceil$, and let $f: V \to \{1,2,\dots,x\}$ be a random vertex-coloring of G, where for each vertex $v \in V$ independently, the color $f(v) \in \{1,2,\dots,x\}$ is chosen randomly according to a uniform distribution on $\{1,2,\dots,x\}$. In order to complete the proof it suffices to show that with positive probability f is an acyclic coloring of G. To this end we define a family of events, apply the local lemma to show that with positive probability none of them occurs, and observe that if none of them occurs then f is acyclic. The events we consider are of the following four types.

a) **Type I:** For each pair of adjacent vertices u and v of G, let $A_{\{u,v\}}$ be the event that $f(u)=f(v)$.

b) **Type II:** For each induced path of length four $v_0v_1v_2v_3v_4$ in G, let $B_{\{v_0,v_1,v_2,v_3,v_4\}}$ be the event that $f(v_0)=f(v_2)=f(v_4)$ and $f(v_1)=f(v_3)$.

A pair of nonadjacent vertices of G is called a *special pair* if they have more than $d^{2/3}$ common neighbors.

c) **Type III:** For each induced 4-cycle $v_1v_2v_3v_4$ in G, in which neither $\{v_1, v_3\}$ nor $\{v_2, v_4\}$ is a special pair, let $C_{\{v_1,v_2,v_3,v_4\}}$ be the event that $f(v_1)=f(v_3)$ and $f(v_2)=f(v_4)$.

d) **Type IV:** For each special pair of vertices u, w in G let $D_{\{u,w\}}$ be the event that $f(u)=f(w)$.

Now suppose that none of the events of the four types above occurs. We claim that f must be an acyclic coloring. Indeed since no events of type I occur, f is a proper coloring. Therefore every odd cycle of G contains vertices of at least three distinct colors. What about the possibility of having an even cycle

$$C = v_0v_1v_2v_3 \cdots v_{2k-1}v_{2k} = v_0$$

in G in which $f(v_0)=f(v_2)=\cdots=f(v_{2k-2})$ and $f(v_1)=\cdots f(v_{2k-1})$? We may restrict our attention to induced cycles, since if we have a diagonal then either we have an odd cycle and hence three colors must occur or we have a smaller two-colored even cycle. Now, induced two-colored 4-cycles do not exist, since no event of type III or IV occurs. Induced two-colored even cycles of length at least six do not exist, since no event of type II occurs. Therefore, if none of the events of types I, II, III, or IV occurs, then f is acyclic, as claimed.

It remains to show that with positive probability none of these events happen. To prove this we apply the local lemma. Let us construct a graph H whose nodes are all the events of all the four types, in which two nodes X_S and Y_T (where $X, Y \in \{A, B, C, D\}$) are adjacent if and only if $S \cap T \neq \phi$. Since the occurrence of each event X_S (for $X \in \{A, B, C, D\}$) depends only on the color of the vertices in S, H is a dependency graph for our events, because even if the colors of all vertices of G but those in S are known, the probability of X_S remains unchanged. Let us call a node of H a type i node, where $i \in \{I, II, III, IV\}$, if it corresponds to an event of type i. In order to apply the local lemma we need an estimate for the number of nodes of each type in H which are adjacent to any given node. This estimate is given in the following two simple lemmas.

Lemma 2.4. *Let v be an arbitrary vertex of the graph $G=(V, E)$. Then the following four statements hold.*

(i) v belongs to at most d edges of G.

(ii) v belongs to at most $3d^4$ induced paths of length 4 in G.
(iii) The number of induced 4-cycles in G containing v in which no opposite pair of vertices is a special pair is at most $d^{8/3}$.
(iv) The number of special pairs of vertices containing v is at most $d^{4/3}$.

Proof. Part (i) is trivial, since $\Delta(G) = d$.

Part (ii) follows from the fact that since $\Delta(G) = d$ the number of paths of length 4 in which v is an end-vertex is at most $d(d-1)^3 \leq d^4$, the number of paths of length 4 in which v is the middle vertex is at most $\binom{d}{2}(d-1)^2 \leq d^4$, and the number of paths of length 4 in which v is a second or fourth vertex is at most $d(d-1)(d-1)^2 \leq d^4$.

To prove (iii) observe that there are at most $d(d-1) \leq d^2$ induced paths of length two vv_1v_2 starting at v. Each induced four-cycle containing v must contain such a path (in fact, it must contain two of them), and if $\{v, v_2\}$ is not a special pair, then there are at most $d^{2/3}$ four-cycles containing the path vv_1v_2. Thus, altogether, there are at most $d^2 \frac{d^{2/3}}{2} < d^{8/3}$ induced 4-cycles containing v, in which no pair of opposite vertices is special.

Part (iv) follows from the fact that there are at most $d(d-1) \leq d^2$ induced paths of length 2 starting at v and more than $d^{2/3}$ of them lead to any vertex u for which $\{u, v\}$ is a special pair. Thus the number of such vertices u is at most $\frac{d^2}{d^{2/3}} = d^{4/3}$. ∎

Lemma 2.5. *For $i, j \in \{I, II, III, IV\}$ the (i, j) entry of the matrix M given below is an upper bound on the number of nodes of type j in the dependency graph H which are adjacent to a node of type i in H.*

M:	I	II	III	IV
I	$2d$	$6d^4$	$2d^{8/3}$	$2d^{4/3}$
II	$5d$	$15d^4$	$5d^{8/3}$	$5d^{4/3}$
III	$4d$	$12d^4$	$4d^{8/3}$	$4d^{4/3}$
IV	$2d$	$6d^4$	$2d^{8/3}$	$2d^{4/3}$

Proof. Let us prove, for example, that the first row of M gives upper bounds for the number of nodes of each type which are adjacent to a type I node. The proofs for the other rows are analogous. Let $A_{\{u,w\}}$ be an event of type I corresponding to a type I node of H. By the definition of H $A_{\{u,w\}}$ is adjacent in H to all type I nodes $A_{z,t}$ where either $u \in \{z, t\}$ or $w \in \{z, t\}$. By Lemma 2.4, part (i) there are at most $2d$ such nodes. Similarly, $A_{\{u,w\}}$ is adjacent in H to all type II nodes $B_{\{v_0,v_1,v_2,v_3,v_4\}}$ where either $u \in \{v_0, \ldots, v_4\}$ or $w \in \{v_0, \ldots, v_4\}$. By Lemma 2.4, part (ii), there are at most $6d^4$ such type II nodes. In the same way, Lemma 2.4 part (iii) implies that at most $2d^{8/3}$ type III nodes are adjacent in H to $A_{\{u,w\}}$ and Lemma 2.4 part (iv) gives that at most $2d^{4/3}$ type IV nodes are adjacent to $A_{\{u,w\}}$. ∎

Recall that $f: V \to \{1, 2, \ldots, x\}$ is a random vertex-coloring of G, where $x = \lceil 50d^{4/3} \rceil$. The following statement is obvious.

Fact 2.6.

(i) For each type I event A, $\Pr(A) = \frac{1}{x}$.
(ii) For each type II event B, $\Pr(B) = \frac{1}{x^3}$
(iii) For each type III event C, $\Pr(C) = \frac{1}{x^2}$
(iv) For each type IV event D, $\Pr(D) = \frac{1}{x}$. ∎

The only ingredient that is still missing for applying the local lemma is the definition of the weights y_i. We define the weight y of each event to be twice its probability, i.e., $\frac{2}{x}$ for events of type I and IV, $\frac{2}{x^3}$ for events of type II and $\frac{2}{x^2}$ for events of type III. By the local lemma (Lemma 2.1) and by Lemma 2.5 and Fact 2.6, in order to conclude that with positive probability none of the forbidden events hold it suffices to verify the following three inequalities.

$$\frac{1}{x} \le \frac{2}{x}\left(1-\frac{2}{x}\right)^{2d+2d^{4/3}}\left(1-\frac{2}{x^3}\right)^{6d^4}\left(1-\frac{2}{x^2}\right)^{2d^{8/3}} \tag{1}$$

$$\frac{1}{x^3} \le \frac{2}{x^3}\left(1-\frac{2}{x}\right)^{5d+5d^{4/3}}\left(1-\frac{2}{x^3}\right)^{15d^4}\left(1-\frac{2}{x^2}\right)^{5d^{8/3}} \tag{2}$$

$$\frac{1}{x^2} \le \frac{2}{x^2}\left(1-\frac{2}{x}\right)^{4d+4d^{4/3}}\left(1-\frac{2}{x^3}\right)^{12d^4}\left(1-\frac{2}{x^2}\right)^{4d^{8/3}}. \tag{3}$$

(Inequality (1) corresponds to events of type I and IV, inequality (2) to events of type II, and inequality (3) to events of type III.) Clearly, the validity of inequality (2) implies that of the other two, and this inequality is valid, since

$$\begin{aligned}&\left(1-\frac{2}{x}\right)^{5d+5d^{4/3}}\left(1-\frac{2}{x^3}\right)^{15d^4}\left(1-\frac{2}{x^2}\right)^{5d^{8/3}}\\ &\ge\left(1-\frac{20d^{4/3}}{x}\right)\left(1-\frac{30d^4}{x^3}\right)\left(1-\frac{10d^{8/3}}{x^2}\right)\\ &\ge\left(1-\frac{20}{50}\right)\left(1-\frac{30}{50^3}\right)\left(1-\frac{10}{50^2}\right)>\frac{1}{2}.\end{aligned}$$

Therefore, with positive probability, f is an acyclic coloring of G. This completes the proof of Proposition 2.2 (and hence that of Theorem 1.1). ∎

Next we prove Theorem 1.2, which shows that the estimate in Theorem 1.1 is not far from being best possible.

Proof of Theorem 1.2. Let $V = \{1, 2, \ldots, n\}$ be a set of n labelled vertices, where 4 divides n. Put

$$p = c\left(\frac{\log n}{n}\right)^{1/4},$$

where $c > 0$ is a constant, independent of n, to be chosen later, and let $G = G_{n,p} = (V, E)$ be a random graph on V obtained by choosing each pair of distinct members of V independently to be an edge with probability p. Let d be the maximum degree on G. By standard estimates on binomial distributions (or by well known results about the degrees of random graphs—see, for example, [4])

$$\Pr\{d \le 2cn^{3/4}(\log n)^{1/4}\} \to 1 \text{ as } n \to \infty .$$

To complete the proof we show that (for an appropriate choice of c)

$$\Pr\left\{A(G) > \frac{n}{2}\right\} \to 1 \text{ as } n \to \infty .$$

To show this we first prove:

Claim 2.7. *For any fixed partition of V into $r \le \frac{n}{2}$ disjoint color classes, the probability that this partition is an acyclic coloring of G is at most $(1-p^4)^{\binom{n/4}{2}}$. (In fact, with probability $1-(1-p^4)^{\binom{n/4}{2}}$ there is a two-colored 4-cycle in G.)*

Proof of Claim 2.7. Let $V_1, V_2, \ldots, V_r$ be the parts of the partition. By omitting a point from each V_i of odd cardinality we obtain at least $n - r \ge \frac{n}{2}$ vertices that lie in disjoint even parts. By partitioning each of these even parts of size >2 vertices into disjoint parts of size 2 we obtain $k = \frac{n}{4}$ pairwise disjoint subsets $U_1 \ldots U_k$ of V, each of cardinality 2, and each a subset of some color class in the original partition. For each $1 \le i < j \le k$, the four edges joining a vertex of U_i to a vertex of U_j would form a 4-cycle. The probability that this 4-cycle is not in G is $1-p^4$ and since all these $\binom{n/4}{2}$ events are mutually independent, as the corresponding 4-cycles are edge-disjoint, the assertion of the claim follows. ■

Returning to the proof of the theorem, observe that there are less than n^n partitions of V. Therefore, the probability that there is an acyclic vertex-coloring with at most $\frac{n}{2}$ colors does not exceed

$$\begin{aligned} n^n(1-p^4)^{\binom{n/4}{2}} &< n^n \exp\left\{-\binom{n/4}{2}p^4\right\} \\ &= \exp\left\{n \log n - \binom{n/4}{2}c^4(\log n)/n\right\}. \end{aligned}$$

This probability is $o(1)$ (as $n \to \infty$) for any fixed c satisfying $c^4 > 32$ (e.g., $c = 3$ will do). This completes the proof. ■

3. GRAPH WITH NO $K_{2,\gamma+1}$

In this section we prove Theorem 1.3, in the following explicit form.

Proposition 3.1. *Let G be a graph with maximum degree $d \ge 1$ and suppose that for some $\gamma \ge 1$, G contains no copy of $K_{2,\gamma+1}$ in which the two vertices in the first class are nonadjacent. Then*

$$A(G) \leq \lceil 32\sqrt{\gamma}d \rceil .$$

We need the following simple lemma.

Lemma 3.2. *Let G be a graph with maximum degree d containing no complete bipartite subgraph $K_{2,\gamma+1}$ in which the two vertices in the first class are nonadjacent. Then, for any $l \geq 4$ and any vertex v of G, the number of induced cycles of length l in G containing v does not exceed $\frac{1}{2}\gamma d^{l-2}$.*

Proof. The number of simple paths of length $l-2$ starting at v and ending in another vertex is at most $d(d-1)^{l-3} \leq d^{l-2}$. Each such path ending at, say, a vertex u, where u is not adjacent to v, can be completed to an induced cycle of length l in G in at most γ k distinct ways, since otherwise u and v would have at least $\gamma + 1$ common neighbors in G, contradicting the hypothesis. Moreover, in this manner each induced cycle of length l containing v is counted twice. This completes the proof. ∎

Proof of Proposition 3.1. Let $G = (V, E)$ be a graph satisfying the assumptions of the proposition. Put $c = 32$, $x = \lceil c\sqrt{\gamma}d \rceil$. We will show that $A(G) \leq x$. Let $f : V \rightarrow \{1, 2, \ldots, x\}$ be a random vertex-coloring of G, where for each vertex $v \in V$ independently, the color $f(v) \in \{1, 2, \ldots, x\}$ is chosen randomly according to a uniform distribution. For each pair of adjacent vertices u and v of G, let $A_{\{u,v\}}$ be the event that $f(u) = f(v)$. We call such an event an event of type A_3. (It is convenient for us to call the type A_3 rather than the more natural A_2). Similarly, for each induced even cycle C of G whose vertices are $v_1, v_2, \ldots, v_{2k}$ (in this cyclic order), let A_C be the event that $f(v_1) = f(v_3) = \ldots = f(v_{2k-1})$ and $f(v_2) = f(v_4) = \ldots = f(v_{2k})$. We call such an event an event of type A_{2k}. As argued in the proof of Theorem 1.1, it is easily seen that if none of these events holds, then f is an acyclic coloring of G. Therefore, in order to complete the proof it suffices to show that with positive probability none of the events above occur. This fact will be proved by applying the local lemma. Let H be the graph whose nodes are all the events $A_{\{u,v\}}$ and A_C, in which two nodes representing two of our events are adjacent if and only if the sets of vertices of G corresponding to the two events share at least one common vertex. Obviously, H is a dependency graph for the events considered, since the occurrence of each $A_C(A_{\{u,v\}})$ depends only on the colors of the vertices of C (the colors of u, v, respectively.)

Every vertex v of G appears in at most d pairs of adjacent vertices $\{u, v\}$. By Lemma 3.2, it appears in at most $\frac{1}{2}\gamma d^{l-2}$ induced cycles of length l, for any even $l \geq 4$. It follows that in the dependency graph H constructed above, for all $k \geq 3$, each event of type A_k is adjacent to at most kd events of type A_3 and to at most $\frac{1}{2}k\gamma d^{l-2}$ events of type A_l $(l \geq 4)$. Therefore (since $\gamma \geq 1$), the following statement holds.

Fact 3.3. In the dependency graph H, for each $k \geq 3$ each event of type A_k is adjacent to at most $k(\sqrt{\gamma}d)^{l-2}$ events of type A_l for all $l \geq 3$.

Clearly, the probability of each event of type A_3 is $\frac{1}{x}$ and that of each event of type A_k is $(\frac{1}{x})^{k-2}$ for all $k \geq 4$. Thus we have:

Fact 3.4. If A is an event of type A_k, then

$$\Pr(A) = \left(\frac{1}{x}\right)^{k-2}.$$

In order to apply the local lemma, we have to define the weights y_i appearing in its statement. For each event A of the A_k define

$$y_A = \frac{c^{\frac{k-2}{2}}}{x^{k-2}},$$

where $c = 32$ is the constant defined in the beginning of the proof and $x = \lceil c\sqrt{\gamma}d \rceil$. Note that for each A, $0 < y_A < 1$. Combining the local lemma (= Lemma 2.1), Fact 3.3, and Fact 3.4 we see that in order to complete the proof it suffices to check that for every $k \geq 3$

$$\frac{1}{x^{k-2}} \leq \frac{c^{\frac{k-2}{2}}}{x^{k-2}} \prod_{l \geq 3} \left(1 - \frac{c^{\frac{l-2}{2}}}{x^{l-2}}\right)^{k(\sqrt{\gamma}d)^{l-2}};$$

that is, that

$$\prod_{l \geq 3} \left(1 - \frac{c^{\frac{l-2}{2}}}{x^{l-2}}\right)^{k\gamma^{\frac{l-2}{2}} d^{l-2}} \geq \frac{1}{c^{\frac{k-2}{2}}}. \tag{4}$$

Since $x = \lceil c\sqrt{\gamma}d \rceil \geq c\sqrt{\gamma}d$ we have

$$1 - \frac{c^{\frac{l-2}{2}}}{x^{l-2}} \geq 1 - \frac{1}{c^{\frac{l-2}{2}} \gamma^{\frac{l-2}{2}} d^{l-2}}.$$

Also, as $c = 32$, $\gamma \geq 1$, $d \geq 1$ we conclude that

$$c^{\frac{l-2}{2}} \gamma^{\frac{l-2}{2}} d^{l-2} > 2,$$

and since $(1 - \frac{1}{z})^z \geq \frac{1}{4}$ for all real $z \geq 2$ the following holds:

$$\begin{aligned}
\prod_{l \geq 3} \left(1 - \frac{c^{\frac{l-2}{2}}}{x^{l-2}}\right)^{k\gamma^{\frac{l-2}{2}} d^{l-2}} &\geq \prod_{l \geq 3} \left(1 - \frac{1}{c^{\frac{l-2}{2}} \gamma^{\frac{l-2}{2}} d^{l-2}}\right)^{k\gamma^{\frac{l-2}{2}} d^{l-2}} \\
&\prod_{l \geq 3} \left(\frac{1}{4}\right)^{k(1/c)^{\frac{l-2}{2}}} \\
&= \left(\frac{1}{4}\right)^{k\Sigma_{l \geq 3}(1/c)^{\frac{l-2}{2}}} > \left(\frac{1}{4}\right)^{\frac{2k}{\sqrt{c}}}.
\end{aligned}$$

It follows that inequality (4) holds provided

$$\left(\frac{1}{4}\right)^{\frac{2k}{\sqrt{c}}} \geq \left(\frac{1}{c}\right)^{\frac{k-2}{2}}$$

for all $k \geq 3$; that is, provided $\sqrt{c}\log_2 c \geq 8\frac{k}{k-2}$ for all $k \geq 3$. Since the maximum of the quantity $8\frac{k}{k-2}$ for $k \geq 3$ is 24 it suffices to check that $\sqrt{c}\log_2 c \geq 24$ and this certainly holds for our choice $c = 32$. We have thus proved that $A(G) \leq \lceil 32\sqrt{\gamma}d \rceil$, completing the proof of Proposition 3.1, and thus of Theorem 1.3. ∎

An edge-coloring of $G = (V, E)$ is called *acyclic* if it is a proper edge coloring (that is, no two incident edges have the same color) and there is no cycle in the subgraph containing all the edges of any two of the colors. The *acyclic edge chromatic number* of G, denoted by $A'(G)$, is the least number of colors in an acyclic edge coloring of G.

For a prime $p > 2$ let K_p denote the complete graph on the p vertices $\{0, 1, \ldots, p-1\}$. Define an edge coloring f of K_p by $f(\{x, y\}) = (x + y) \pmod{p}$. One can easily check that f is an acyclic edge coloring of K_p and hence $A'(K_p) \leq p$. Also $A'(K_p) \geq p$ since any matching in K_p contains at most $\frac{p-1}{2}$ edges. Hence $A'(K_p) = p$.

A similar construction can be used to show that for every prime $p > 2$, the complete bipartite graph $K_{p-1,p-1}$ has $A'(K_{p-1,p-1}) = p$. (Denote the $p-1$ vertices in each of the two color classes X and Y by $1, 2, \ldots, p-1$ and color the edge $\{x, y\}$ for $x \in X$ and $y \in Y$ by $(x + y) \pmod{p}$ to obtain an acyclic edge-coloring, showing that $A'(K_{p-1,p-1}) \leq p$. To show that $A'(K_{p-1,p-1}) \geq p$ observe that any matching in $K_{p-1,p-1}$ is of size at most $p-1$, and if we have two matchings of size $p-1$, then their union contains a cycle.

By known results about the distribution of primes (see, for example, [9]) the above observations show that $A'(K_n) = n + O(n^{2/3})$ and $A'(K_{n,n}) = n + O(n^{2/3})$ as $n \to \infty$.

Define $A'(d) = \max\{A'(G) : \Delta(G) = d\}$. Since each acyclic vertex coloring of the line graph of a graph G gives an acyclic edge coloring of G, and since a line graph contains no copy of $K_{2,5}$ in which the two vertices in the first class are nonadjacent, Theorem 1.3 implies that $A'(d) = O(d)$. This is precisely the assertion of Corollary 1.4.

4. CONCLUDING REMARKS AND OPEN PROBLEMS

1) Theorem 1.3 implies that for every fixed $d \geq 1$ a random d-regular graph $G_{n,d}$ on n vertices satisfies, almost surely (that is, with probability that tends to 1 as n tends to infinity) $A(G_{n,d}) = O(d)$. Similarly, for every fixed $\epsilon > 0$ a random graph $G_{n,p}$ obtained by taking n labelled vertices and choosing each pair of them to be an edge randomly and independently with probability p, where $p \leq 1n^{\frac{1}{2}+\epsilon}$, satisfies, almost surely $A(G_{n,p}) = O(\Delta(G_{n,p})) + 1$.

2) The probabilistic construction given in the proof of Theorem 1.2 shows that the estimate given in Theorem 1.1 is not far from being sharp. This construction gives, almost surely, a graph G with maximum degree d, which contains no copy of $K_{2,\gamma+1}$ for some $\gamma = O(d^{2/3}(\log d)^{1/3})$ and still satisfies $A(G) = \Gamma(d^{4/3}/$

$(\log d)^{1/3})$. Thus, the estimate in Theorem 1.3 is sharp here up to a logarithmic factor.

3) Consider a graph G with maximum degree d. Let us call a coloring of G 'P_k-free' if as usual no two adjacent vertices have the same color and also no path P_k with k vertices is 2-colored. Thus a P_4-free coloring must be acyclic. The earlier proof that $A(G) \leq d^2 + 1$ in fact shows that there is always a P_3-free coloring with at most this number of colors. For such colorings we may need $(1 + o(1))d^2$ colors. For whenever q is a prime power, there is a graph G with $n = q^2 + q + 1$ vertices and maximum degree $q + 1$, which has diameter 2 and thus has $A(G) = n$ (see, for example, [5, page 176]).

If in the proof of Proposition 2.2 we consider only events of the first two types, then we see easily that G must have a P_5-free coloring with $O(d^{4/3})$ colors. A similar proof shows that for any fixed $k \geq 5$ there is always a P_k-free coloring with $O(d^{\frac{k-1}{k-2}})$ colors.

4) It would be interesting to close the gap in the estimates given for $A(d)$ by Theorems 1.1 and 1.2. We suspect that the upper bound, given in Theorem 1.1, is closer to the truth.

5) We do not have any explicit construction of graphs G for which $A(G) \gg \Delta(G)$ (the existence of such graphs is proved in Theorem 1.2). Such a construction, besides being interesting in its own right, may help to improve the lower bound for $A(d)$ given in Theorem 1.2, as it frequently happens that probabilistic arguments of the type used in the proof of this theorem supply estimates that deviate by some logarithmic factors from the best possible estimates.

6) The proofs of Theorems 1.1 and 1.3 are not constructive, and since they apply the local lemma they do not supply even efficient randomized algorithms for the corresponding problems. It would be interesting to find a polynomial time (deterministic or randomised) algorithm that would find, given a graph G with maximum degree d, an acyclic coloring of it using $O(d^{4/3})$ (or even $o(d^2)$) colors, or an acyclic edge coloring of it, using $O(d)$ colors.

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